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ON UNIQUENESS OF THE DYNAMIC FINITE-STEP PROBLEM IN GRADIENT-DEPENDENT SOFTENING PLASTICITY

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Abstract—The dynamic evolution of an elastoplastic softening solid is considered. A material model including in the yield function the Laplacian of the plastic multiplier is used to regularize the problem. The dynamic finite-step problem is formulated according to a generalized mid-point integration scheme. Space discretization is carried out by a mixed finite element technique based on generalized variables. A sufficient uniqueness condition of the finite-step solution is proved. For a one-dimensional problem also a necessary and sufficient condition is presented. A simple numerical test shows the regularizing properties (mesh-independence) of the proposed model and the positive influence of the gradient term also on the time step amplitude ensuring uniqueness of solution. Copyright © 1996 Published by Elsevier Science Ltd.

1. INTRODUCTION

As nowadays is well known, the initial boundary value problems for classical softening elastoplastic or damaging continua can become ill-posed and various bifurcation phenomena such as strain localization may occur. These phenomena show up both in static and in dynamic situations. Among many representative contributions we only quote here those of Maier (1969), (1971), Hill and Hutchinson (1975), Rice (1976), and the recent critical survey by Bažant and Cedolin (1991).

In numerical analyses the use of classical continuum softening models results in pathological mesh-dependence [Bažant (1976), de Borst (1987)] and possible non uniqueness of the incremental solution [Maier and Perego (1992), Comi *et al.* (1992a)].

Many approaches have been proposed in order to obtain mesh objectivity; all of them imply the introduction of a characteristic internal length of the material which governs the localized behaviour. Among various modifications intended to regularize the *classical* softening continuum we quote here the introduction of a viscous term [see e.g. Loret and Prevost (1990) for plasticity and Dubé *et al.* (1994) for damage models], the polar continua formulation [de Borst and Sluys (1991), Fleck and Hutchinson (1993), Steinmann and Stein (1994)], the non-local models [Pijaudier-Cabot and Bažant (1987)], the gradient-dependent models [Schreyer and Chen (1986), Mühlhaus and Aifantis (1991), de Borst and Mühlhaus (1992), Aifantis (1992), Sluys *et al.* (1993), Benallal and Tvergaard (1995)].

The present paper deals with the dynamic evolution of an elastoplastic softening continuum in the small deformation range, regularized by the introduction in the yield function of the Laplacian of the equivalent plastic strain. In particular we focus on properties concerning the dynamic finite-step problem, i.e. the algebraic set of relations governing the dynamic response of the structure in a single time-step after discretization in time and space. The issue of uniqueness of the finite-step solution is addressed and sufficient conditions are proved.

Uniqueness of incremental solutions is an important aspect of evolutive analyses, since it rules out bifurcation and alternative dynamic scenarios of the kind pointed out by Maier and Perego (1992). Sufficient uniqueness conditions for the dynamic finite-step problem have already been proposed in Comi *et al.* (1992a) with reference to *classical* (non regularized) softening plasticity.

In finite element analyses involving gradient dependent models, plastic consistency cannot be enforced locally at the Gauss point level and *ad hoc* finite-element procedures must be developed. In this paper, the finite element formulation is derived making use of the so-called *generalized variables* approach, as proposed in Comi and Perego (1995b) in the quasi static context. This amounts to a variationally consistent way of modeling in space the unknown fields, including the internal variables which govern irreversible processes, thus obtaining a *constitutive law* for the whole finite element aggregate.

The notion of *generalized variables*, introduced by Prager (1952) with reference to beam problems, was extensively developed in the context of piece-wise linear plasticity by Maier (1968), De Donato and Maier (1972), and Corradi (1978, 1983, 1986). More recently, in Comi *et al.* (1992b) and Comi and Perego (1995a) generalized variable formulations have been proposed for the standard internal variable material models of Halphen and Nguyen (1975).

An outline of the paper is as follows. The equations governing the dynamic evolution of an elastoplastic body are presented in Section 2, for a softening gradient-dependent material model. Time discretization is introduced according to the generalized mid-point integration rule proposed in Corigliano and Perego (1993). A mixed variational formulation for the governing equations discretized in time is then presented; it extends to dynamics the one obtained in Comi and Perego (1995b, 1996). The variational formulation represents the basis for the introduction of spatial discretization in terms of generalized variables which allows us to obtain the relations governing the dynamic finite-step.

Section 3 concerns some algorithmic aspects relevant to the proposed formulation; in particular the choice of interpolations and the iterative solution procedure adopted to solve the finite-step problem are discussed.

In Section 4 sufficient conditions for uniqueness of the finite-step solution are proved for Mises and Drucker-Prager models, with isotropic hardening and second order gradient term. One of these conditions provides directly an upper bound on the admissible time-step.

In order to show the effectiveness of the generalized variables approach and the regularizing effect of gradient-dependent models, numerical examples are presented in Section 5, where a one-dimensional bar problem in pure tension is analyzed. The mesh objectivity is shown and a discussion on uniqueness is presented. For this simple example also a necessary and sufficient condition for uniqueness is obtained and compared with the sufficient ones of Section 4.

2. FORMULATION AND DISCRETIZATION OF THE DYNAMIC PROBLEM FOR GRADIENT-DEPENDENT PLASTICITY

2.1. *Continuum problem*

Let us consider the dynamic evolution of an elasto-plastic softening body of volume Ω and surface Γ in the small strain range. Dynamic equilibrium, initial conditions and compatibility are expressed by the following set of equations:

$$\mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}(\mathbf{x},t)+\mathbf{F}(\mathbf{x},t) = \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x},t) \quad \text{in } \Omega; \quad \mathbf{n}(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{f}(\mathbf{x},t) \quad \text{on } \Gamma_f \tag{1}$$

$$\mathbf{u}(\mathbf{x},t_0) = \mathbf{u}_0(\mathbf{x}); \quad \dot{\mathbf{u}}(\mathbf{x},t_0) = \dot{\mathbf{u}}_0(\mathbf{x}) \quad \text{in } \Omega \tag{2}$$

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \mathbf{C}\mathbf{u}(\mathbf{x},t) \quad \text{in } \Omega; \quad \mathbf{u}(\mathbf{x},t) = \mathbf{U}(\mathbf{x},t) \quad \text{on } \Gamma_u \tag{3}$$

Equations (1) express dynamic equilibrium in volume Ω and on the free part of the boundary Γ_f. For simplicity, viscous damping effects are neglected. Vector $\boldsymbol{\sigma}$ contains six independent components of the symmetric Cauchy stress tensor; $\mathbf{C}^{\mathrm{T}}$ is the differential operator of equilibrium in matrix form ($\mathbf{C}$ being the operator of linear compatibility); $\mathbf{F}$ and $\mathbf{f}$ are body and surface loads respectively; ρ is the mass density; $\mathbf{u}$ is the three component displacement vector; a superimposed dot means derivation with respect to time; $\mathbf{n}$ is a matrix whose entries are the components of the outward normal to the boundary Γ. The initial conditions on displacements and velocities are given by eqns (2). Linear compatibility in volume Ω and on the constrained part of the boundary Γ_u are described by eqns (3), where vector $\boldsymbol{\varepsilon}$ gathers the six independent components of the small strain tensor and $\mathbf{U}$ is the vector of assigned displacements. Notice that, in order to maintain the scalar product and to easily define the second invariant of the stress or strain tensors, vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are defined as follows:

$$\boldsymbol{\sigma}^{\mathrm{T}} \equiv [\sigma_x\ \sigma_y\ \sigma_z\ \sqrt{2}\sigma_{xy}\ \sqrt{2}\sigma_{xz}\ \sqrt{2}\sigma_{yz}]\,; \quad \boldsymbol{\varepsilon}^{\mathrm{T}} \equiv [\varepsilon_x\ \varepsilon_y\ \varepsilon_z\ \sqrt{2}\varepsilon_{xy}\ \sqrt{2}\varepsilon_{xz}\ \sqrt{2}\varepsilon_{yz}] \tag{4}$$

As a consequence, $\mathbf{C}$ has the following expression:

$$\mathbf{C}^{\mathrm{T}} \equiv \begin{bmatrix} \partial_x & 0 & 0 & \frac{1}{\sqrt{2}}\partial_y & \frac{1}{\sqrt{2}}\partial_z & 0 \\ 0 & \partial_y & 0 & \frac{1}{\sqrt{2}}\partial_x & 0 & \frac{1}{\sqrt{2}}\partial_z \\ 0 & 0 & \partial_z & 0 & \frac{1}{\sqrt{2}}\partial_x & \frac{1}{\sqrt{2}}\partial_y \end{bmatrix} \tag{5}$$

where ∂_ξ denote the partial derivative operator with respect to ξ.

The material behaviour is governed by a standard elastoplastic associative model with linear hardening or softening, modified by the introduction in the yield condition of the second order gradient of the plastic multiplier, as proposed e.g. by Mühlhaus and Aifantis (1991), de Borst and Mühlhaus (1992), Sluys *et al.* (1993).

$$\boldsymbol{\varepsilon}(\mathbf{x},t) = \mathbf{e}(\mathbf{x},t)+\mathbf{p}(\mathbf{x},t)\,; \quad \boldsymbol{\sigma}(\mathbf{x},t) = \mathbf{E}(\mathbf{x})\mathbf{e}(\mathbf{x},t) \quad \text{in}\ \Omega \tag{6}$$

$$\Phi(\mathbf{x},t) = \varphi(\boldsymbol{\sigma}(\mathbf{x},t)) - h(\mathbf{x})\lambda(\mathbf{x},t) + c(\mathbf{x})\nabla^2\lambda(\mathbf{x},t) - k(\mathbf{x}) \leqslant 0\,; \quad \dot{\lambda} \geqslant 0\,; \quad \Phi\dot{\lambda} = 0 \quad \text{in}\ \Omega \tag{7}$$

$$[(\nabla\dot{\lambda}(\mathbf{x},t))^{\mathrm{T}}\mathbf{m}(\mathbf{x})]\dot{\lambda}(\mathbf{x},t) = 0 \quad \text{on}\ \Gamma_p \tag{8}$$

$$\dot{\mathbf{p}}(\mathbf{x},t) = \partial_\sigma\varphi(\boldsymbol{\sigma}(\mathbf{x},t))\dot{\lambda}(\mathbf{x},t) \quad \text{in}\ \Omega \tag{9}$$

Equation (6a) expresses additivity of elastic $\mathbf{e}$ and plastic $\mathbf{p}$ strains, while eqn (6b) expresses linear elasticity. Due to the definition of vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the matrix of elastic moduli $\mathbf{E}$ acquires the following form:

$$\mathbf{E} \equiv \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & \text{sym} & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \tag{10}$$

where λ and μ are Lame's constants.

The elastic domain and the loading-unloading conditions are defined by eqns (7a–d); $\varphi(\boldsymbol{\sigma})$ is an equivalent stress, h is the constant isotropic hardening/softening parameter, λ is the plastic multiplier, c is a non-negative diffusion parameter with the dimension of a force, and k is the initial yield limit. If $c = 0$, the non standard term containing the Laplacian of λ disappears and the classical plasticity model is recovered. In the following we will consider linear softening behaviour, i.e. $h < 0$. The equivalent stress $\varphi(\boldsymbol{\sigma})$ is assumed to be given by a convex, order one homogeneous function of stresses; in particular we will consider the Mises equivalent stress $\varphi(\boldsymbol{\sigma}) = \sqrt{3J_2} = \sqrt{3/2\,\mathbf{s}^T\mathbf{s}}$ and the Drucker-Prager equivalent stress $\varphi(\boldsymbol{\sigma}) = \sqrt{3J_2} + \alpha I_1/3$, where vector **s** gathers the six independent components of the deviator stress tensor defined analogously at $\boldsymbol{\sigma}$ in eqn (4a), J_2 is the second invariant of the deviator stress tensor, I_1 the first invariant of the stress tensor and α the friction coefficient. Equation (8) expresses the additional boundary conditions, in the form proposed by Mühlhaus and Aifantis (1991), which are required to complete the problem due to the presence of the gradient term ($\nabla^2\lambda$) in the yield function, **m** is the outward normal vector to the boundary Γ_p of the plastic zone. These so called *ambiguous boundary conditions* are fulfilled if either the plastic multiplier rate or its gradient in the normal direction are zero on the boundary Γ_p. Equation (9) gives the associative flow rule for the plastic strains.

As shown by Mühlhaus and Aifantis (1991), the present gradient-dependent model can be interpreted as a simplified version of fully nonlocal models in which the stress at a position **x** depends on the average strain within some symmetric neighbourhood of **x** defined by a characteristic length of the material. The micro-mechanical grounds of the inclusion of gradients in the constitutive model are discussed e.g. by Aifantis (1987, 1992).

The gradient-dependent plasticity formulation is apt to simulate shear banding and localization phenomena exhibited by both ductile and quasi-brittle materials. The inclusion of gradients in softening or damage constitutive laws makes it possible to describe not only the inception of localization, but also the post-localization behaviour.

The regularizing effects due to the introduction in the yield condition of the Laplacian of the plastic multiplier, with a positive coefficient c, have been discussed by several authors [see e.g. de Borst and Mühlhaus (1992), Mühlhaus and Aifantis (1991), Sluys *et al.* (1993), Benallal and Tvergaard (1995)]. In the dynamic context, Sluys *et al.* (1993) have shown that, in the presence of a second-order gradient term, the wave equation remains hyperbolic and the initial value problem well-posed also in the softening regime. Furthermore, wave propagation remains dispersive and the phase velocity remains real if the wave length is greater than a material-dependent internal length which is related to the diffusion parameter c.

2.2. *Time discretization*

Let $t_0, t_1, \ldots, t_n, t_{n+1} = t_n + \Delta t, \ldots$ be convenient time instants along the time interval over which the dynamic response of the body is sought. Consider the time step $\Delta t = t_{n+1} - t_n$: at $t = t_n$ all quantities are known and the solution must be computed at t_{n+1} for given load increments $\Delta\mathbf{F}$ and $\Delta\mathbf{f}$. The time-integrated version of the dynamic initial-boundary value problem (1)–(9) is obtained by adopting distinct mid-point approximations for velocities and accelerations, by enforcing dynamic equilibrium at the end of the step and by making use of a backward difference scheme for the constitutive law:

$$\Delta\mathbf{u} = \Delta t\,\dot{\mathbf{u}}_{n+\beta} = \Delta t[(1-\beta)\dot{\mathbf{u}}_n + \beta\dot{\mathbf{u}}_{n+1}]; \quad \Delta\dot{\mathbf{u}} = \Delta t\,\ddot{\mathbf{u}}_{n+\gamma} = \Delta t[(1-\gamma)\ddot{\mathbf{u}}_n + \gamma\ddot{\mathbf{u}}_{n+1}] \tag{11}$$

$$\mathbf{C}^T\boldsymbol{\sigma}_{n+1} + \mathbf{F}_{n+1} = \rho\ddot{\mathbf{u}}_{n+1} \quad \text{in } \Omega; \quad \mathbf{n}\boldsymbol{\sigma}_{n+1} = \mathbf{f}_{n+1} \quad \text{on } \Gamma_f \tag{12}$$

$$\mathbf{C}\,\Delta\mathbf{u} = \Delta\boldsymbol{\varepsilon} = \Delta\mathbf{e} + \Delta\mathbf{p} \quad \text{in } \Omega; \quad \Delta\mathbf{u} = \Delta\mathbf{U} \quad \text{on } \Gamma_u; \quad \Delta\boldsymbol{\sigma} = \mathbf{E}\,\Delta\mathbf{e} \quad \text{in } \Omega \tag{13}$$

$$\Phi_{n+1} = \varphi(\boldsymbol{\sigma}_{n+1}) - h\lambda_{n+1} + c\nabla^2\lambda_{n+1} - k \leqslant 0; \quad \Delta\lambda \geqslant 0; \quad \Phi_{n+1}\,\Delta\lambda = 0 \quad \text{in } \Omega \tag{14}$$

$$[(\nabla\lambda_{n+1})^T\mathbf{m}]\,\Delta\lambda = 0 \quad \text{on } \Gamma \tag{15}$$

$$\Delta\mathbf{p} = \partial_\sigma\varphi(\boldsymbol{\sigma}_{n+1})\,\Delta\lambda \quad \text{in } \Omega \tag{16}$$

where subscripts n, $n+1$, $n+\beta$ and $n+\gamma$ ($\beta, \gamma \in]0, 1]$) mark quantities at t_n, t_{n+1}, $t_{n+\beta}$ and $t_{n+\gamma}$ respectively, $\Delta\bullet$ denotes the increment of the quantity $\bullet$ over Δt and the explicit dependence of all quantities on **x** has been omitted for notation convenience.

From eqns (11) it follows that:

$$\ddot{\mathbf{u}}_{n+1} = \frac{1}{\gamma\beta\,\Delta t^2}\mathbf{u}_{n+1} - \left[\frac{1-\gamma}{\gamma}\ddot{\mathbf{u}}_n + \frac{1}{\gamma\beta\,\Delta t}\dot{\mathbf{u}}_n + \frac{1}{\gamma\beta\,\Delta t^2}\mathbf{u}_n\right] \tag{17}$$

The introduction of eqn (17) into the equation of motion (12a) yields:

$$\mathbf{C}^{\mathrm{T}}\boldsymbol{\sigma}_{n+1} + \mathbf{F}^{*}_{n+1} = \rho^{*}\mathbf{u}_{n+1} \text{ in } \Omega \tag{18}$$

having set

$$\rho^{*} \equiv \frac{1}{\gamma\beta\,\Delta t^2}\rho;\quad \mathbf{F}^{*}_{n+1} \equiv \mathbf{F}_{n+1} + \rho\left[\frac{1-\gamma}{\gamma}\ddot{\mathbf{u}}_n + \frac{1}{\gamma\beta\,\Delta t}\dot{\mathbf{u}}_n + \frac{1}{\gamma\beta\,\Delta t^2}\mathbf{u}_n\right] \tag{19}$$

The above approximate time integration scheme belongs to the class of generalized midpoint dynamic algorithms proposed in Corigliano and Perego (1993), where a discussion on non-linear stability properties has been presented together with other features of the algorithm. When $\beta = \gamma$, the above class of algorithms coincides with the sub-class of the Newmark family identified by Newmark parameters $\gamma^{*} = 2\beta^{*}$. In particular by choosing $\beta = \gamma = 1/2$ (i.e. $\beta^{*} = 1/4, \gamma^{*} = 1/2$), one obtains the average acceleration method. Another noteworthy choice is $\beta = \gamma = 0.6$, this gives an algorithm with damping properties at the higher frequencies which will be used in the numerical examples of Section 5.

As done in the quasi-static case [see Comi and Perego (1995b, 1996)], it is possible to build a variational formulation of the dynamic finite-step problem (12b)–(16), (18). To this purpose let us define the functional $\mathscr{L}$:

$$\begin{aligned}\mathscr{L} \equiv &\int_{\Omega}\frac{1}{2}\rho^{*}\mathbf{u}^{\mathrm{T}}_{n+1}\mathbf{u}_{n+1}\,\mathrm{d}\Omega + \int_{\Omega}\frac{1}{2}\mathbf{e}^{\mathrm{T}}_{n+1}\mathbf{E}\mathbf{e}_{n+1}\,\mathrm{d}\Omega + \int_{\Omega}\boldsymbol{\sigma}^{\mathrm{T}}_{n+1}(\mathbf{C}\,\Delta\mathbf{u} - \Delta\mathbf{e})\,\mathrm{d}\Omega \\ &- \int_{\Omega}(\varphi(\boldsymbol{\sigma}_{n+1}) - k)\,\Delta\lambda\,\mathrm{d}\Omega + \int_{\Omega}\frac{1}{2}h\lambda^2_{n+1}\,\mathrm{d}\Omega + \int_{\Omega}\frac{1}{2}c(\nabla\lambda_{n+1})^{\mathrm{T}}(\nabla\lambda_{n+1})\,\mathrm{d}\Omega \\ &- \int_{\Omega}\mathbf{F}^{*\mathrm{T}}_{n+1}\,\Delta\mathbf{u}\,\mathrm{d}\Omega - \int_{\Gamma_t}\mathbf{f}^{\mathrm{T}}_{n+1}\,\Delta\mathbf{u}\,\mathrm{d}\Gamma\end{aligned} \tag{20}$$

A solution of the following saddle-point problem:

$$\min_{\Delta\mathbf{u},\Delta\mathbf{e},\Delta\lambda}\ \max_{\Delta\sigma}\{\mathscr{L}\} \quad \text{with } \Delta\lambda \in \mathscr{K} \equiv \{\Delta\lambda \mid \Delta\lambda \geqslant 0 \quad \text{in } \Omega\}$$

$$\Delta\mathbf{u} \in \mathscr{U} \equiv \{\Delta\mathbf{u} \mid \Delta\mathbf{u} = \Delta\mathbf{U} \quad \text{on } \Gamma_u\} \tag{21}$$

is a solution of the dynamic finite-step problem. The converse in general does not hold true since the functional is not convex in $\Delta\lambda$ when softening is present. The proof of the above statement follows the same path of the one presented in Comi and Perego (1996) for a quasi-static evolution and is not duplicated here for brevity. We only recall here that the minimization of (20) with respect to $\Delta\lambda \in \mathscr{K}$ gives rise to a variational inequality which, after integration by parts, gives the loading-unloading conditions (14) in Ω and the boundary conditions (15) on Γ. Therefore the additional boundary conditions involving $\Delta\lambda$ are part of the solution of problem (21) and need not be included in the definition of the admissible set $\mathscr{K}$. It is also worth noticing that conditions (15) are not directly the time integrated version of condition (8).

2.3. *Finite element discretization in terms of generalized variables*

A finite element formulation of the dynamic finite-step problem can be obtained by introducing in the functional $\mathscr{L}$ independent modeling of all fields:

$$\mathbf{u}^h(\mathbf{x},t) = \mathbf{N}_u(\mathbf{x})\bar{\mathbf{u}}(t), \quad \mathbf{e}^h(\mathbf{x},t) = \mathbf{N}_e(\mathbf{x})\bar{\mathbf{e}}(t), \tag{22a, b}$$

$$\boldsymbol{\sigma}^h(\mathbf{x},t) = \mathbf{N}_\sigma(\mathbf{x})\bar{\boldsymbol{\sigma}}(t), \quad \Delta\lambda^h(\mathbf{x},t) = \mathbf{N}_\lambda(\mathbf{x})\,\Delta\bar{\lambda}(t) \tag{22c,d}$$

where $\mathbf{N}_\alpha$ are matrices containing the interpolation functions for the field α and barred quantities are independent parameters. All vectors and matrices are to be intended as referring to the whole finite element aggregate.

If the interpolations of conjugate fields $\mathbf{e}$ and $\boldsymbol{\sigma}$ are chosen in a way which guarantees conservation of the scalar product, the independent parameters can be interpreted as *generalized variables* in Prager's sense [see e.g. Corradi (1986) and Comi *et al.* (1992b) for details]:

$$\bar{\boldsymbol{\sigma}}(t)^{\mathrm{T}}\bar{\mathbf{e}}(t) = \int_\Omega [\boldsymbol{\sigma}^h(\mathbf{x},t)]^{\mathrm{T}}\mathbf{e}^h(\mathbf{x},t)\,\mathrm{d}\Omega = \bar{\boldsymbol{\sigma}}(t)^{\mathrm{T}}\int_\Omega \mathbf{N}_\sigma(\mathbf{x})^{\mathrm{T}}\mathbf{N}_e(\mathbf{x})\,\mathrm{d}\Omega\,\bar{\mathbf{e}}(t) \Leftrightarrow \int_\Omega \mathbf{N}_\sigma(\mathbf{x})^{\mathrm{T}}\mathbf{N}_e(\mathbf{x})\,\mathrm{d}\Omega = \mathbf{I} \tag{23}$$

By also approximating the convex cone $\mathscr{K}$ by $\bar{\mathscr{K}}$ and $\mathscr{U}$ by $\bar{\mathscr{U}}$

$$\bar{\mathscr{K}} = \{\Delta\lambda^h = \mathbf{N}_\lambda\,\Delta\bar{\boldsymbol{\lambda}} \mid \Delta\bar{\boldsymbol{\lambda}} \geqslant \mathbf{0}\}\,; \quad \bar{\mathscr{U}} = \{\Delta\mathbf{u}^h = \mathbf{N}_u\,\Delta\bar{\mathbf{u}} \mid \Delta\bar{\mathbf{u}} = \Delta\bar{\mathbf{U}} \text{ on } \Gamma_u\} \tag{24}$$

the discretized form of the optimization problem (21) reads:

$$\min_{\Delta\bar{\mathbf{u}},\Delta\bar{\mathbf{e}},\Delta\bar{\boldsymbol{\lambda}}}\ \max_{\Delta\bar{\boldsymbol{\sigma}}}\{\mathscr{L}\} \quad \text{with} \quad \Delta\bar{\boldsymbol{\lambda}}\in\bar{\mathscr{K}}, \quad \Delta\bar{\mathbf{u}}\in\bar{\mathscr{U}} \tag{25}$$

and

$$\begin{aligned}\mathscr{L} = {} & \int_\Omega \frac{1}{2}\rho^*\bar{\mathbf{u}}^{\mathrm{T}}\mathbf{N}_u^{\mathrm{T}}\mathbf{N}_u\bar{\mathbf{u}}\,\mathrm{d}\Omega + \int_\Omega \frac{1}{2}\bar{\mathbf{e}}^{\mathrm{T}}\mathbf{N}_e^{\mathrm{T}}\mathbf{E}\mathbf{N}_e\bar{\mathbf{e}}\,\mathrm{d}\Omega + \int_\Omega \bar{\boldsymbol{\sigma}}^{\mathrm{T}}\mathbf{N}_\sigma^{\mathrm{T}}(\mathbf{C}\mathbf{N}_u\,\Delta\bar{\mathbf{u}} - \mathbf{N}_e\,\Delta\bar{\mathbf{e}})\,\mathrm{d}\Omega \\ & - \int_\Omega (\varphi(\mathbf{N}_\sigma\bar{\boldsymbol{\sigma}}) - k)\mathbf{N}_\lambda\,\Delta\bar{\boldsymbol{\lambda}}\,\mathrm{d}\Omega + \int_\Omega \frac{1}{2}h\bar{\boldsymbol{\lambda}}^{\mathrm{T}}\mathbf{N}_\lambda^{\mathrm{T}}\mathbf{N}_\lambda\bar{\boldsymbol{\lambda}}\,\mathrm{d}\Omega + \int_\Omega \frac{1}{2}c\bar{\boldsymbol{\lambda}}^{\mathrm{T}}(\nabla\mathbf{N}_\lambda)^{\mathrm{T}}\nabla\mathbf{N}_\lambda\bar{\boldsymbol{\lambda}}\,\mathrm{d}\Omega \\ & - \int_\Omega \mathbf{F}^{*\mathrm{T}}\mathbf{N}_u\,\Delta\bar{\mathbf{u}}\,\mathrm{d}\Omega - \int_{\Gamma_f}\mathbf{f}^{\mathrm{T}}\mathbf{N}_u\,\Delta\bar{\mathbf{u}}\,\mathrm{d}\Gamma\end{aligned} \tag{26}$$

In (26), and from now onward, quantities at the end of the step, i.e. at t_{n+1}, are denoted without subscript, e.g. $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}_n + \Delta\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}_{n+1}$.

The governing relations of the discrete dynamic finite step problem result from the Kuhn-Tucker optimality conditions of problem (25):

$$\bar{\mathbf{L}}\bar{\mathbf{u}} + \bar{\mathbf{C}}^{\mathrm{T}}\bar{\boldsymbol{\sigma}} = \bar{\mathbf{F}} \tag{27}$$

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{E}}\bar{\mathbf{e}} \tag{28}$$

$$\bar{\boldsymbol{\Phi}} \equiv \bar{\boldsymbol{\varphi}}(\bar{\boldsymbol{\sigma}}) - (\bar{\mathbf{h}} + \bar{\mathbf{c}})\bar{\boldsymbol{\lambda}} - \bar{\mathbf{k}} \leqslant \mathbf{0}, \quad \Delta\bar{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \bar{\boldsymbol{\Phi}}^{\mathrm{T}}\,\Delta\bar{\boldsymbol{\lambda}} = 0 \tag{29}$$

$$\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}} = \Delta\bar{\mathbf{e}} + \partial_\sigma\bar{\boldsymbol{\varphi}}^{\mathrm{T}}(\bar{\boldsymbol{\sigma}})\,\Delta\bar{\boldsymbol{\lambda}} \equiv \Delta\bar{\mathbf{e}} + \Delta\bar{\mathbf{p}} \tag{30}$$

where the following definitions of generalized quantities have been introduced:

$$\bar{\mathbf{L}} = \frac{\mathbf{M}}{\beta\gamma\,\Delta t^2} \equiv \int_\Omega \rho^*\mathbf{N}_u^{\mathrm{T}}\mathbf{N}_u\,\mathrm{d}\Omega\,; \quad \bar{\mathbf{C}} \equiv \int_\Omega \mathbf{N}_\sigma^{\mathrm{T}}\mathbf{C}\mathbf{N}_u\,\mathrm{d}\Omega\,; \quad \bar{\mathbf{F}} \equiv \int_\Omega \mathbf{N}_u^{\mathrm{T}}\mathbf{F}^*\,\mathrm{d}\Omega + \int_{\Gamma_f}\mathbf{N}_u^{\mathrm{T}}\mathbf{f}\,\mathrm{d}\Gamma \tag{31}$$

$$\bar{\mathbf{E}} \equiv \int_\Omega \mathbf{N}_e^{\mathrm{T}} \mathbf{E} \mathbf{N}_e \, \mathrm{d}\Omega ; \quad \bar{\boldsymbol{\varphi}} \equiv \int_\Omega \mathbf{N}_\lambda^{\mathrm{T}} \varphi(\mathbf{N}_\sigma \bar{\boldsymbol{\sigma}}) \, \mathrm{d}\Omega ; \quad \bar{\mathbf{h}} \equiv \int_\Omega h \mathbf{N}_\lambda^{\mathrm{T}} \mathbf{N}_\lambda \, \mathrm{d}\Omega \tag{32}$$

$$\bar{\mathbf{c}} \equiv \int_\Omega c(\nabla \mathbf{N}_\lambda)^{\mathrm{T}} \nabla \mathbf{N}_\lambda \, \mathrm{d}\Omega ; \quad \bar{\mathbf{k}} \equiv \int_\Omega \mathbf{N}_\lambda^{\mathrm{T}} k \, \mathrm{d}\Omega ; \quad \Delta \bar{\mathbf{p}} \equiv \partial_\sigma \bar{\boldsymbol{\varphi}}^{\mathrm{T}} \Delta \bar{\lambda} \tag{33}$$

In the equilibrium eqn (27) and in eqn (31a), matrix $\bar{\mathbf{L}}$ is the algorithmic mass matrix and $\mathbf{M}$ is the consistent mass matrix of the finite element aggregate. For numerical convenience the consistent mass matrix is often substituted by a lumped version.

The set of eqns (28)–(30) can be interpreted as a *constitutive law* for the whole discretized structure. The gradient term of the continuum model gives rise to the matrix $\bar{\mathbf{c}}$ which is summed in eqn (29a) to the hardening/softening matrix $\bar{\mathbf{h}}$. It is important to remark that, due to the different structure of matrices $\bar{\mathbf{h}}$ and $\bar{\mathbf{c}}$, $\bar{\mathbf{c}}$ cannot be simply interpreted as an additional hardening matrix. The presence of $\bar{\mathbf{c}}$ in eqn (29a) allows for the regularizing effects as it will be shown later.

Remark. Numerical solution of problems dealing with *enhanced non-local* constitutive laws require the development of specific finite element formulations since the constitutive law cannot be enforced locally at the Gauss point level as done in the usual displacement based finite element method. The present generalized variable approach gives a variationally consistent way to obtain such a formulation in which also the inelastic variables are modelled and it naturally leads to a *non-local discrete* constitutive law.

3. ALGORITHMIC ASPECTS

3.1. *Choice of interpolations*

As usual in finite element modeling, some requirements concerning the choice of interpolation functions must be satisfied *a priori*. Since first-order spatial derivatives of displacements and of the plastic multiplier appear in the functional $\mathscr{L}$ of eqn (20), the interpolation functions $\mathbf{N}_u(\mathbf{x})$ and $\mathbf{N}_\lambda(\mathbf{x})$ must be at least C^0 continuous in Ω, i.e. also across elements. As a consequence, the interpolation parameters $\bar{\mathbf{u}}$ and $\Delta\bar{\lambda}$ most naturally acquire the meaning of values of the corresponding approximated fields $\mathbf{u}^h(\mathbf{x}, t)$ and $\Delta\lambda^{\mathrm{h}}(\mathbf{x}, t)$ at the n_n nodes of the structure. It is worth noting that, while the continuity requirement on $\mathbf{N}_u(\mathbf{x})$ is usual in displacement-based finite element formulations, the continuity of $\mathbf{N}_\lambda(\mathbf{x})$ is here originated by the gradient term in the constitutive law. For classical plasticity laws, plastic multipliers could also be modeled as discontinuous across elements. Moreover, in order to be consistent with the variational formulation (25), function $\bar{\boldsymbol{\varphi}}$ must be convex. From definition (32b) of $\bar{\boldsymbol{\varphi}}$, this entails further mild restrictions on the choice of $\mathbf{N}_\lambda(\mathbf{x})$ which will be assumed to be satisfied in the following.

As far as $\mathbf{N}_e(\mathbf{x})$ and $\mathbf{N}_\sigma(\mathbf{x})$ are concerned, the only requirement is given by the *orthogonality condition* (23). A possible and convenient choice, proposed in Corradi (1986) and in Comi *et al.* (1992b) and which will be assumed in the following, consists of modeling elastic strains and stresses in terms of their values at the Gauss points. These points are the same as those used to numerically compute integrals in eqns (31)–(33). Considering for instance a function $f(\mathbf{x})$, the integral over Ω is computed in the following way:

$$\int_\Omega f(\mathbf{x}) \, \mathrm{d}\Omega \cong \sum_{k=1}^{n_e} \sum_{j=1}^{n_{gp}} f(\mathbf{x}^j) w^j |\det J^k| \equiv \sum_{g=1}^{n_{Gp}} f(\mathbf{x}^g) W^g \tag{34}$$

where n_e is the number of elements; n_{gp} is the number of Gauss points per element; $n_{Gp} = n_{gp} \times n_e$ is the number of Gauss points in the whole structure; $\mathbf{x}^j$ and w^j are the coordinates and the weight of the j-th Gauss point; $\det J^k$ is the Jacobian of the iso-parametric mapping of element k. The weight W^g, pertinent to the g-th Gauss point in the global numeration, is defined by (34b).

With the above definitions, the following interpolations for elastic strains are then assumed:

$$\mathbf{N}_e(\mathbf{x}) \equiv [diag(N_G^1(\mathbf{x}))\dots diag(N_G^g(\mathbf{x}))\dots diag(N_G^{n_{Gp}}(\mathbf{x}))]\,; \quad g = 1\dots n_{Gp} \tag{35}$$

where $N_G^g(\mathbf{x})$ are polynomial functions which vanish in all Gauss points except from the g-th where they have unit value and $diag(N_G^g)$, $g = 1 \dots n_{Gp}$ are diagonal matrices of order equal to the number of the local independent strain components.

As a consequence of the above interpolation, the generalized variable vector $\bar{\mathbf{e}}$ collects n_{Gp} sub vectors coinciding with the local approximated elastic strains $\mathbf{e}^h$ evaluated at the n_{Gp} Gauss points.

The stresses are modeled by:

$$\mathbf{N}_\sigma(\mathbf{x}) \equiv \left[\frac{diag(N_G^1(\mathbf{x}))}{W^1}\dots\frac{diag(N_G^g(\mathbf{x}))}{W^g}\dots\frac{diag(N_G^{n_{Gp}}(\mathbf{x}))}{W^{n_{Gp}}}\right] \quad g = 1\dots n_{Gp} \tag{36}$$

Therefore, the vector of generalized stresses $\bar{\boldsymbol{\sigma}}$ collects n_{Gp} sub-vectors which coincide with the local approximated stresses $\boldsymbol{\sigma}^h$ evaluated at the n_{Gp} Gauss points multiplied by the corresponding Gauss weight W.

The above choices for $\mathbf{N}_e$ and $\mathbf{N}_\sigma$ satisfy the condition (23c), as can be verified by making use of eqns (34)–(36). Moreover, the matrix of generalized elastic moduli $\bar{\mathbf{E}}$ turns out to be block diagonal and the resulting *generalized Hooke's law* (28) decouples at the Gauss point level, i.e. it can be imposed at each of the n_{Gp} Gauss points separately.

On the contrary, the continuity of $\mathbf{N}_\lambda$ across elements implies that the matrices $\bar{\mathbf{h}}$ and $\bar{\mathbf{c}}$ exhibit a banded but *non-diagonal* pattern. Therefore the n_n generalized yield modes $\bar{\Phi}_\alpha$ depend on the n_n generalized plastic multipliers $\bar{\lambda}_\beta$ in a coupled way, the plastic part of the generalized constitutive law involves simultaneously all the finite elements and cannot be imposed separately at specific points.

For later use, let us also define the *diagonal* softening matrix $\hat{\mathbf{h}}$:

$$\hat{\mathbf{h}} \equiv \operatorname*{diag}_g\,[\hat{\mathbf{h}}^g]\,; \quad \hat{\mathbf{h}}^g = \Theta h(\mathbf{x}^g)W^g\mathbf{I}\,; \quad g = 1\dots n_{Gp} \tag{37a, b}$$

with

$$\Theta \equiv \begin{cases} \dfrac{2}{3} & \text{Mises model} \\[2ex] \dfrac{6}{9+2\alpha^2} & \text{Drucker-Prager model} \end{cases} \tag{37c}$$

where W^g is the Gauss weight defined in eqn (34b) and $\mathbf{I}$ is the identity matrix of order equal to the number of the local independent strain components.

In classical non-gradient plasticity full decoupling of the constitutive law at the Gauss point level can be achieved by modeling the plastic multiplier in terms of its value at the Gauss points by the same interpolation functions N_G^g used for the strains (eqn 35) [see Comi and Perego (1995a)].

To satisfy the stability condition of Babuska-Brezzi (Babuska (1973), Brezzi (1974)) in the elastic range one must assume a number of generalized stresses and strains which is not less than the number of the free nodal displacements and ensure that $\bar{\mathbf{C}}\bar{\mathbf{u}} \neq \mathbf{0}$ for any $\bar{\mathbf{u}} \neq \mathbf{0}$ (see Zienkiewicz *et al.* (1986)). These conditions can be easily fulfilled by the interpolation (35) and (36) assuming a suitable number of Gauss points for each element, for instance the number which would allow for a full integration of the stiffness matrix. It should be noted that in this case the limitation principle proved in Stolarski and Belyschko (1987) holds and the generalized variable approach, in the elastic range, turns out to be coincident with the displacement formulation.

3.2. *Solution procedure for the finite-step problem*

The nonlinear dynamic finite-step problem (27)–(30) can be solved iteratively according to a Newton-Raphson or a modified Newton-Raphson scheme.

Each iteration can be divided into a *predictor phase* providing an estimate of displacement increments on the basis of linearized equilibrium equations and a *corrector phase* providing stresses which fulfill the constitutive law for assigned total strain increments.

Predictor phase—at the iteration $i+1$ compute an estimate $\bar{\mathbf{u}}^{i+1}$ of displacements at t_{n+1} from:

$$\bar{\mathbf{u}}^{i+1} = \bar{\mathbf{u}}^{i} - (\mathbf{S}^{i+1})^{-1}(\bar{\mathbf{L}}\bar{\mathbf{u}} + \bar{\mathbf{C}}^{\mathrm{T}}\bar{\boldsymbol{\sigma}} - \bar{\mathbf{F}})^{i} \tag{38}$$

where $\mathbf{S}^{i+1}$ is a suitably chosen prediction matrix. In the following, $\mathbf{S}^{i+1}$ will be assumed equal to the elastic predictor $\bar{\mathbf{L}}+\mathbf{K}_E$, $\mathbf{K}_E = \bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{E}}\bar{\mathbf{C}}$, for each step and iteration.

Corrector phase—considering $\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^{i+1} = \bar{\mathbf{C}}(\bar{\mathbf{u}}^{i+1} - \bar{\mathbf{u}}_n)$ as the driving quantity, solve the constitutive law checking first the plastic activation:

(a) if $\bar{\Phi}_\alpha^{trial} \equiv \bar{\varphi}_\alpha(\bar{\boldsymbol{\sigma}}_n + \bar{\mathbf{E}}\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^{i+1}) - ((\bar{\mathbf{h}}+\bar{\mathbf{c}})\bar{\boldsymbol{\lambda}}_n)_\alpha - \bar{k}_\alpha > 0$ for at least one $\alpha \in [1, n_n]$, then solve eqns (29) with: $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}_n + \bar{\mathbf{E}}(\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^{i+1} - \partial_{\bar{\sigma}}\bar{\boldsymbol{\varphi}}^{\mathrm{T}}(\bar{\boldsymbol{\sigma}})\,\Delta\bar{\boldsymbol{\lambda}})$

(b) if $\bar{\Phi}_\alpha^{trial} \leqslant 0$ for every α, then $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}_n + \bar{\mathbf{E}}\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^{i+1}$ and $\Delta\bar{\boldsymbol{\lambda}} = \mathbf{0}$.

What is peculiar about the generalized variable formulation for gradient plasticity considered here is that not only the linear prediction, but also the nonlinear correction must be carried out at the global level. Moreover, the non linearity of the equations governing the corrector phase requires the use of an *ad hoc* iterative scheme, in addition to the global Newton-Raphson procedure.

It is important to note that, for a class of material models including the Mises and the Drucker-Prager models, it is possible to express the effective stress $\varphi(\boldsymbol{\sigma})$ as a linear function of $\Delta\lambda$: $\varphi(\boldsymbol{\sigma}) = \varphi^{trial} + a\,\Delta\lambda$, a being a constant. With the adopted interpolations for stresses and plastic multiplier this property holds also for the generalized effective stress. The nonlinear corrector phase can thus be recast in the form of a standard Linear Complementarity Problem in the unknowns $\Delta\bar{\boldsymbol{\lambda}}$:

$$\mathbf{q} + \mathbf{D}\,\Delta\bar{\boldsymbol{\lambda}} \geqslant \mathbf{0}; \quad \Delta\bar{\boldsymbol{\lambda}} \geqslant \mathbf{0}; \quad \Delta\bar{\boldsymbol{\lambda}}^{\mathrm{T}}(\mathbf{q} + \mathbf{D}\,\Delta\bar{\boldsymbol{\lambda}}) = 0 \tag{39}$$

where $\mathbf{q}$ is a constant vector and $\mathbf{D}$ is a constant matrix whose definitions depend on the effective stress.

The linear complementarity problem can be solved by standard mathematical programming algorithms; in the numerical simulations of Section 5 we adopt the Mangasarian's scheme [Mangasarian (1977)].

4. UNIQUENESS CRITERIA FOR THE FINITE STEP SOLUTION

In the presence of softening behaviour, for an assigned discretization in space, uniqueness of the solution for the dynamic finite-step problem is not guaranteed. In this section we prove sufficient conditions for uniqueness of the finite step solution in terms of displacements increments and plastic strain increments. These conditions are proved under the following assumptions, which particularize the formulation of Section 2:

(a) the effective stress $\varphi(\boldsymbol{\sigma})$ is chosen equal to the Mises function: $\varphi(\boldsymbol{\sigma}) = \sqrt{3J_2}$ or the Drucker-Prager function: $\varphi(\boldsymbol{\sigma}) = \sqrt{3J_2} + \alpha I_1/3$;

(b) interpolations of the independent fields are chosen as specified in Section 3.1; namely stresses and elastic strains are modeled in terms of their values at Gauss points, displacements and plastic multiplier are modeled in terms of nodal values and the interpolations of plastic multiplier are such as to guarantee convexity of the generalized effective stress $\bar{\boldsymbol{\varphi}}$;

(c) a positive definite matrix $\bar{\mathbf{L}}$ is assumed, deriving through eqn (31a) from a consistent or a positive definite lumped mass matrix.

Proposition 1. Given the initial state at t_n and the external load vector at t_{n+1}, the dynamic finite-step solution is unique in terms of $\Delta\bar{\mathbf{u}}$ and $\Delta\bar{\mathbf{p}}$ (and hence of $\Delta\bar{\mathbf{e}}$ and $\Delta\bar{\boldsymbol{\sigma}}$), if the matrix:

$$\mathbf{G} \equiv \begin{bmatrix} \bar{\mathbf{L}}+\mathbf{K}_E & -\bar{\mathbf{C}}^T\bar{\mathbf{E}} \\ -\bar{\mathbf{E}}\bar{\mathbf{C}} & \bar{\mathbf{E}}+\bar{\mathbf{h}} \end{bmatrix} \tag{40}$$

is positive definite.

Proof. Consider two hypothetically different solutions of the dynamic finite-step problem (27)–(30) denoted by superscripts $'$ and $''$; mark their difference with $*$, e.g. $\bar{\mathbf{u}}^* \equiv \bar{\mathbf{u}}'-\bar{\mathbf{u}}'' = \Delta\bar{\mathbf{u}}^*$.

Being $\Delta\bar{\boldsymbol{\sigma}}^* = \bar{\mathbf{E}}\,\Delta\bar{\mathbf{e}}^*$ in equilibrium with $-\bar{\mathbf{L}}\,\Delta\bar{\mathbf{u}}^*$ (eqns 27, 28) and $\Delta\bar{\mathbf{e}}^*+\Delta\bar{\mathbf{p}}^*$ compatible with $\Delta\bar{\mathbf{u}}^*$ (eqn 30), the virtual work principle yields:

$$\Delta\bar{\mathbf{u}}^{*T}\bar{\mathbf{L}}\,\Delta\bar{\mathbf{u}}^*+\Delta\bar{\mathbf{e}}^{*T}\bar{\mathbf{E}}\,\Delta\bar{\mathbf{e}}^*+\Delta\bar{\mathbf{p}}^{*T}\,\Delta\bar{\boldsymbol{\sigma}}^* = 0 \tag{41}$$

The convexity of $\bar{\boldsymbol{\varphi}}(\bar{\boldsymbol{\sigma}})$ and the loading/unloading conditions (29) allow to derive the following inequality:

$$\bar{\boldsymbol{\sigma}}^{*T}\,\Delta\bar{\mathbf{p}}^*-\bar{\boldsymbol{\lambda}}^{*T}(\bar{\mathbf{h}}+\bar{\mathbf{c}})\,\Delta\bar{\boldsymbol{\lambda}}^* \geqslant 0 \tag{42}$$

which is a consequence of Hill's maximum dissipation principle. By making use of (42), eqn (41) can be transformed into the following inequality:

$$\Delta\bar{\mathbf{u}}^{*T}\bar{\mathbf{L}}\,\Delta\bar{\mathbf{u}}^*+\Delta\bar{\mathbf{e}}^{*T}\bar{\mathbf{E}}\,\Delta\mathbf{e}^*+\Delta\bar{\boldsymbol{\lambda}}^{*T}(\bar{\mathbf{h}}+\bar{\mathbf{c}})\,\Delta\bar{\boldsymbol{\lambda}}^* \leqslant 0 \tag{43}$$

The first two addends in the above inequality can be interpreted as twice the *algorithmic* kinetic energy and elastic strain energy in the step, respectively; they are never negative and vanish if and only if $\Delta\bar{\mathbf{u}}^* = \bar{\mathbf{u}}'-\bar{\mathbf{u}}'' = \mathbf{0}$ and $\Delta\bar{\mathbf{e}}^* = \bar{\mathbf{e}}'-\bar{\mathbf{e}}'' = \mathbf{0}$. When matrix $(\bar{\mathbf{h}}+\bar{\mathbf{c}})$ is positive definite, uniqueness of $\bar{\mathbf{u}}$, $\bar{\mathbf{e}}$, $\bar{\boldsymbol{\lambda}}$ follows; when matrix $(\bar{\mathbf{h}}+\bar{\mathbf{c}})$ is positive semidefinite only uniqueness of $\bar{\mathbf{u}}$ and $\bar{\mathbf{e}}$ is guaranteed. In the case here treated of softening behaviour, a sign-definition to matrix $(\bar{\mathbf{h}}+\bar{\mathbf{c}})$ cannot be given *a priori*. Nevertheless the following inequality can be proved (see Appendix A):

$$\Delta\bar{\boldsymbol{\lambda}}^{*T}\bar{\mathbf{h}}\,\Delta\bar{\boldsymbol{\lambda}}^* \geqslant \Delta\bar{\mathbf{p}}^{*T}\bar{\mathbf{h}}\,\Delta\bar{\mathbf{p}}^* \tag{44}$$

Moreover $\bar{\mathbf{c}}$ is a positive semidefinite matrix, since the gradient parameter c in eqn (33a) is non-negative. By taking into account inequality (44), the positive semi-definiteness of $\bar{\mathbf{c}}$ and the compatibility relation (30), inequality (43) gives rise to:

$$\mathscr{F} = \begin{Bmatrix} \Delta\bar{\mathbf{u}}^* \\ \Delta\bar{\mathbf{p}}^* \end{Bmatrix}^T \begin{bmatrix} \bar{\mathbf{L}}+\mathbf{K}_E & -\bar{\mathbf{C}}^T\bar{\mathbf{E}} \\ -\bar{\mathbf{E}}\bar{\mathbf{C}} & \bar{\mathbf{E}}+\bar{\mathbf{h}} \end{bmatrix} \begin{Bmatrix} \Delta\bar{\mathbf{u}}^* \\ \Delta\bar{\mathbf{p}}^* \end{Bmatrix} \leqslant 0 \tag{45}$$

By hypothesis the matrix $\mathbf{G}$ of the above quadratic form $\mathscr{F}$ is positive definite, then from (45) it follows $\Delta\bar{\mathbf{u}}^* = \bar{\mathbf{u}}'-\bar{\mathbf{u}}'' = \mathbf{0}$, $\Delta\bar{\mathbf{p}}^* = \bar{\mathbf{p}}'-\bar{\mathbf{p}}'' = \mathbf{0}$ and uniqueness is proved.

Proposition 2. The dynamic finite-step solution is unique in terms of $\Delta\bar{\mathbf{u}}$ and $\Delta\bar{\mathbf{p}}$ if a scalar $\zeta > 0$ can be found such that:

(a) matrix $\bar{\mathbf{L}}-\zeta\mathbf{K}_E$ is positive definite;

(b) matrix $\zeta/(\zeta+1)\bar{\mathbf{E}}+\hat{\mathbf{h}}$ is positive semidefinite.

Proof. Split the quadratic form in (45) into the following sum of addends:

$$\mathscr{F} = \mathscr{F}_1+\mathscr{F}_2+\mathscr{F}_3 \tag{46}$$

$$\mathscr{F}_1 = \Delta\bar{\mathbf{u}}^{*\mathrm{T}}(\bar{\mathbf{L}}-\zeta\mathbf{K}_E)\,\Delta\bar{\mathbf{u}}^* \tag{47}$$

$$\mathscr{F}_2 = \frac{\Delta\bar{\mathbf{p}}^{*\mathrm{T}}}{\sqrt{1+\zeta}}[(\bar{\mathbf{E}}+\hat{\mathbf{h}})(1+\zeta)-\bar{\mathbf{E}}]\frac{\Delta\bar{\mathbf{p}}^*}{\sqrt{1+\zeta}} \tag{48}$$

$$\mathscr{F}_3 = \left\{\sqrt{1+\zeta}\,\Delta\bar{\mathbf{u}}^{*\mathrm{T}}\ \frac{\Delta\bar{\mathbf{p}}^{*\mathrm{T}}}{\sqrt{1+\zeta}}\right\}\begin{bmatrix}\mathbf{K}_E & -\bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{E}}\\ -\bar{\mathbf{E}}\bar{\mathbf{C}} & \bar{\mathbf{E}}\end{bmatrix}\begin{Bmatrix}\sqrt{1+\zeta}\,\Delta\bar{\mathbf{u}}^*\\ \dfrac{\Delta\mathbf{p}^*}{\sqrt{1+\zeta}}\end{Bmatrix} \tag{49}$$

Note that $\mathscr{F}_3 \geqslant 0$ for any $\sqrt{1+\zeta}\,\Delta\bar{\mathbf{u}}^*$ and $\Delta\bar{\mathbf{p}}^*/\sqrt{1+\zeta}$ since it is twice the elastic energy corresponding to $\Delta\bar{\mathbf{e}}^* = \sqrt{1+\zeta}\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^*-\Delta\bar{\mathbf{p}}^*/\sqrt{1+\zeta}$. By hypotheses, $\mathscr{F}_1$ and $\mathscr{F}_2$ are non-negative. Therefore, to comply with inequality (45), $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}_3$ must vanish separately. Due to the positive-definiteness of $\bar{\mathbf{L}}-\zeta\mathbf{K}_E$, $\mathscr{F}_1 = 0$ implies $\Delta\bar{\mathbf{u}}^* = \mathbf{0}$. $\mathscr{F}_3 = 0$ if and only if $\sqrt{1+\zeta}\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}}^* = \Delta\bar{\mathbf{p}}^*/\sqrt{1+\zeta}$, also $\Delta\bar{\mathbf{p}}^* = \mathbf{0}$, which proves Proposition 2.

Proposition 3. The dynamic finite-step solution is unique in terms of $\Delta\bar{\mathbf{u}}$ and $\Delta\bar{\mathbf{p}}$ if the time step Δt for the adopted integration scheme fulfills the inequality:

$$\Delta t < \frac{1}{\omega_{\mathrm{M}}\sqrt{\zeta\beta\gamma}} \equiv \Delta t_{SC} \tag{50}$$

(ω_{M} being the highest eigenfrequency), with ζ such that the matrix $\zeta/(\zeta+1)\bar{\mathbf{E}}+\hat{\mathbf{h}}$ is positive semidefinite.

Proof. Let $\mathbf{\Psi}$ denote the eigenmode matrix of the structural model supposed to be elastic and let $\mathbf{X}$ denote the principal coordinates such that $\bar{\mathbf{u}} = \mathbf{\Psi}\mathbf{X}$. Equivalent mass, elastic stiffness and eigenfrequency of the j-th mode will be denoted by m_j, k_j and $\omega_j = (k_j/m_j)^{1/2}$.

In view of the definition of $\bar{\mathbf{L}}$ in (31a), the quadratic form $\mathscr{F}_1$ (eqn 47) can be written as:

$$\mathscr{F}_1 = \mathbf{X}^{\mathrm{T}}\left[\mathbf{\Psi}^{\mathrm{T}}\frac{\mathbf{M}}{\beta\gamma\,\Delta t^2}\mathbf{\Psi}-\zeta\mathbf{\Psi}^{\mathrm{T}}\mathbf{K}_E\mathbf{\Psi}\right]\mathbf{X} = \mathbf{X}^{\mathrm{T}}\,\underset{j}{diag}\left[\frac{m_j}{\beta\gamma\,\Delta t^2}(1-\zeta\omega_j^2\beta\gamma\,\Delta t^2)\right]\mathbf{X} \tag{51}$$

The positive definiteness of $\mathscr{F}_1$ is guaranteed if the expression in round brackets is positive for all j, which in turn is guaranteed by hypothesis (50). By proposition 2, uniqueness of the finite-step solution follows.

Remarks

1. Proposition 3 gives the maximum time step amplitude Δt_{SC} ensuring uniqueness. Δt_{SC} depends on the chosen dynamic integration algorithm through parameters β and γ, on the amount of softening through ζ and on the inertia and stiffness of the system through ω_{M}. The information given by proposition 3 is practically useful, in fact in eqn (50) ω_{M}, or an upper bound on this quantity, is cheap to obtain and parameter ζ can be computed at the Gauss point level (once for all Gauss points if h is uniform in the structure) due to the particular structure of $\hat{\mathbf{h}}$ and $\bar{\mathbf{E}}$. A reasonably quick estimate of ω_{M} can be obtained by computing the maximum eigenfrequency of a single element [see e.g. Hughes *et al.* (1979),

Flanagan and Belytschko (1981)]. Propositions 1 and 2 are more difficult to apply since the matrices involved are defined at the structural level.

2. Matrix $\bar{\mathbf{c}}$, coming from the second order gradient term, does not appear explicitly in the above uniqueness conditions; in fact the non-negative term $\Delta\bar{\boldsymbol{\lambda}}^{*T}\bar{\mathbf{c}}\,\Delta\bar{\boldsymbol{\lambda}}^{*}$ has been dropped from (43), preserving the inequality. The positive effect on uniqueness of matrix $\bar{\mathbf{c}}$ comes out by considering that matrix $\bar{\mathbf{h}}+\bar{\mathbf{c}}$ could be positive semidefinite also in the presence of softening. The influence of $\bar{\mathbf{c}}$ on the finite step amplitude Δt_{NSC} necessary and sufficient to guarantee uniqueness will be highlighted in the example of Section 5.

3. Viscous damping could be introduced in the model and uniqueness propositions 1–3 could be proved without additional difficulties as done in Comi *et al.* (1992a) with reference to classical softening plasticity and linear kinematic hardening.

4. If the uniqueness condition of Proposition 1 is fulfilled, the solution of the dynamic finite-step problem is equivalent to the solution of a constrained minimum problem, as in the case of stable materials. On the basis of this extremum principle it is possible to obtain a sufficient criterion for convergence of the iterative predictor-corrector scheme outlined in Section 3.2. Both the extremum property and the convergence criterion can be obtained following the same path of reasoning illustrated in Comi *et al.* (1992a) which is not duplicated here for brevity.

5. ONE-DIMENSIONAL PROBLEM

In order to investigate the regularization properties of the gradient model, to check in this context the soundness of the generalized variable approach and to discuss the uniqueness conditions proved in Section 4, it is here considered the one-dimensional bar problem in pure tension. This has already been used as a reference problem in Sluys *et al.* (1993) and allows for analytical as well as numerical derivations. The data concerning geometry, material and loading conditions are given in Fig. 1.

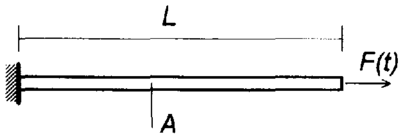
For this one dimensional problem the effective stress is assumed simply as $\varphi(\sigma) = |\sigma|$, which turns out to be linear in $\Delta\lambda$. In order to show this, consider the one-dimensional Hooke's law in the finite-step, E being the Young modulus:

$$\sigma = E(\varepsilon - p_n) - E\frac{\partial\varphi}{\partial\sigma}\Delta\lambda = E(\varepsilon - p_n) - E\frac{\sigma}{|\sigma|}\Delta\lambda \tag{52}$$

From the above relation one obtains:

$$\sigma = \frac{E(\varepsilon - p_n)}{1 + \dfrac{E}{|\sigma|}\Delta\lambda}; \quad |\sigma| = \frac{E|\varepsilon - p_n|}{1 + \dfrac{E}{|\sigma|}\Delta\lambda} \tag{53}$$

Solving eqn (53b) for $|\sigma|$, one has $\varphi(\sigma) = |\sigma| = E|\varepsilon - p_n| - E\Delta\lambda$. As remarked in Section



$L = 100$ mm
$A = 1$ mm^2
$F(t) = \begin{cases} 0 & \text{for } t < 0 \\ 0.75\sigma_0 A & \text{for } t \geq 0 \end{cases}$
$\rho = 2\cdot 10^{-8}$ Ns^2/mm^4
$\sigma_0 = 2$ MPa
$E = 20000$ MPa
$h = -2000$ MPa
$c = 50000$ N

Fig. 1. One dimensional problem: geometrical, loading and material data.

3.2, the linearity of the effective stress in $\Delta\lambda$ allows us to formulate the corrector phase as a linear complementarity problem.

5.1. *Formulation and dynamic response*

We have discretized the bar by truss elements, formulated according to that discussed in Section 3.1, with a constant model for stresses and elastic strains (1 Gauss point per element) and a linear one for displacements and plastic multiplier (two nodes per element). A lumped mass matrix has been assumed. The vectors and matrices concerning the single finite element are shown in box 1, where Ω^e and l^e are the volume and the length of the finite element e, respectively.

$$\mathbf{N}_\epsilon^e = 1\,; \quad \mathbf{N}_\sigma^e = \frac{1}{\Omega^e}; \quad \mathbf{N}_u^e = \mathbf{N}_\lambda^e = \left[\left(1-\frac{x}{l^e}\right)\frac{x}{l^e}\right]; \quad \bar{\mathbf{C}}^e = \left[-\frac{1}{l^e}\;\frac{1}{l^e}\right]; \quad \mathbf{M}^e = \rho\Omega^e\begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$$

$$\bar{\boldsymbol{\varphi}}^e = \frac{1}{2}|\bar{\sigma}^e|\begin{bmatrix}1\\1\end{bmatrix}; \quad \bar{\mathbf{k}}^e = \frac{1}{2}\Omega^e k\begin{bmatrix}1\\1\end{bmatrix}; \quad \bar{\mathbf{h}}^e = \frac{\Omega^e h}{3}\begin{bmatrix}1 & 1/2\\ 1/2 & 1\end{bmatrix}; \quad \bar{\mathbf{c}}^e = \frac{c}{l^{e2}}\Omega^e\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\bar{\mathbf{q}}^e = [(\bar{\mathbf{h}}+\bar{\mathbf{c}})\bar{\boldsymbol{\lambda}}_n]^e + \bar{\mathbf{k}}^e - \frac{1}{2}\Omega^e E[|(\bar{\mathbf{C}}\bar{\mathbf{u}})^e - \mathrm{p}_n^e|]\begin{bmatrix}1\\1\end{bmatrix}; \quad \mathbf{D}^e = \frac{1}{4}\Omega^e E\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + \bar{\mathbf{h}}^e + \bar{\mathbf{c}}^e$$

Box 1. Truss element: generalized variable matrices

Dynamic analyses were performed with 3 different meshes (20, 40, 80 elements), following the iterative scheme proposed in Section 3.2, choosing for the dynamic integration $\beta = \gamma = 0.6$ and assuming an elastic predictor at each iteration (modified Newton-Raphson scheme). Even though the rate of convergence for the discretized residuum is only linear, this scheme was preferred here to the Newton-Raphson one to avoid the computation of the tangent matrix that, due to the non-local character of the constitutive law, would require the inversion of matrices concerning the whole structure (see Comi and Perego (1996)). The time step amplitude has been assumed for all analyses equal to 5×10^{-7} s. As will be discussed later this amplitude ensures uniqueness of the dynamic finite step problem.

When the elastic wave front reaches the built-in end of the structure, plastic strains develop in the element close to this end, due to the reflection of the wave and a localization zone of plastic deformation emerges. The width of the localization zone is constant upon mesh refinement, as shown in Fig. 2; it depends only on the internal length introduced in the model by means of the parameter c [cf. Sluys *et al.* (1993)].

In Fig. 3 the distributions of stresses along the bar, at the end of the time interval, are shown for the various meshes considered.

Figure 4 shows the time history of the reaction at the built-in end of the bar for the different meshes. Figures 4a and 4b concern analyses done with different values of parameters β and γ of the time integration scheme: $\beta = \gamma = 0.6$ in Fig. 4a, $\beta = \gamma = 0.5$ in Fig. 4b. With the first choice the time integration algorithm shows a damping effect for high frequencies, while with the second choice (average acceleration) no damping effect is present. This difference in *algorithmic damping* can be appreciated by comparing the curves in Fig. 4a and 4b. After plastification the value of the reaction oscillates; the oscillation are rapidly damped in the case of Fig. 4a while are much more persistent in the case of Fig. 4b. The damping effect is more appreciated for refined meshes i.e. with higher frequencies. When the spurious oscillations are dumped, the response becomes almost mesh independent due to the regularization effect of the gradient term.

The above results are in complete agreement with those presented in Sluys *et al.* (1993). However, it should be noted that, with the present generalized variable formulation, only first order continuity for interpolation of plastic multiplier is required. Moreover, due to

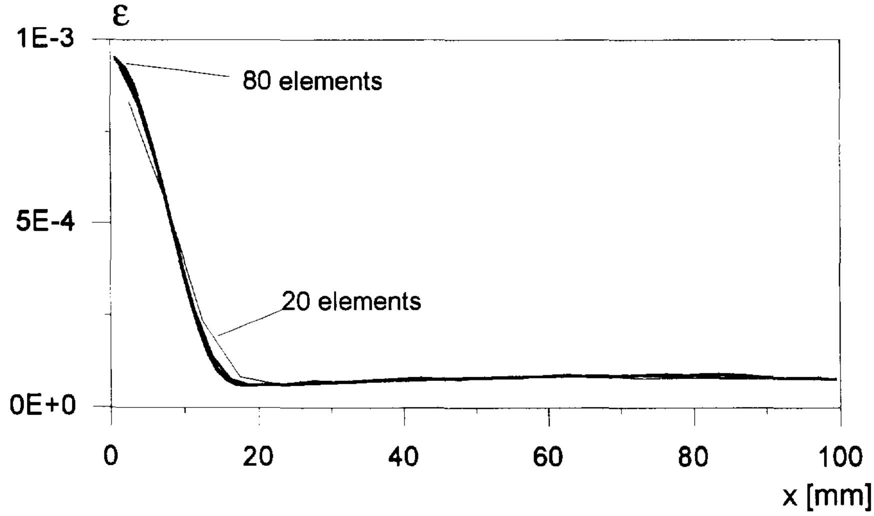


Fig. 2. Strain profile along the bar at $t = 2 \times 10^{-4}$ s for the three meshes.

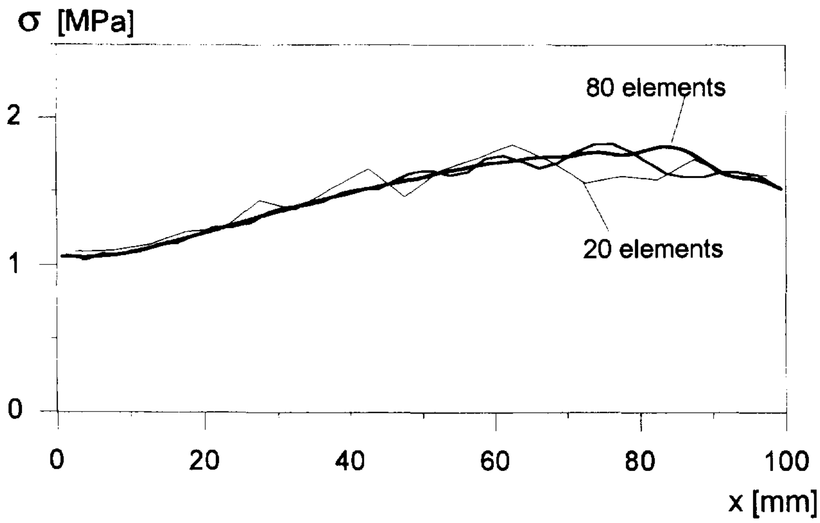


Fig. 3. Stress profile along the bar at $t = 2 \times 10^{-4}$ s for the three meshes.

the variational basis of the proposed discretization, no boundary conditions need to be imposed to the generalized plastic multipliers.

5.2. *Discussion on uniqueness*

For the simple problem in point, it is possible to formulate the whole finite-step problem as a linear complementarity problem, similarly to what done in the works of Maier [see e.g. Maier (1968, 1970)]. In fact, for the one dimensional problem in pure tension the effective stress $\varphi(\sigma)$ is simply equal to σ and from (32b) the generalized effective stress $\bar{\varphi}$ becomes:

$$\bar{\boldsymbol{\varphi}} = \int_\Omega \mathbf{N}_\lambda^T \mathbf{N}_\sigma \bar{\boldsymbol{\sigma}}\, d\Omega = \bar{\mathbf{N}}^T \bar{\boldsymbol{\sigma}}, \quad \bar{\mathbf{N}} \equiv \int_\Omega \mathbf{N}_\sigma^T \mathbf{N}_\lambda\, d\Omega \tag{54}$$

Thus the yield condition (29a) reduces to:

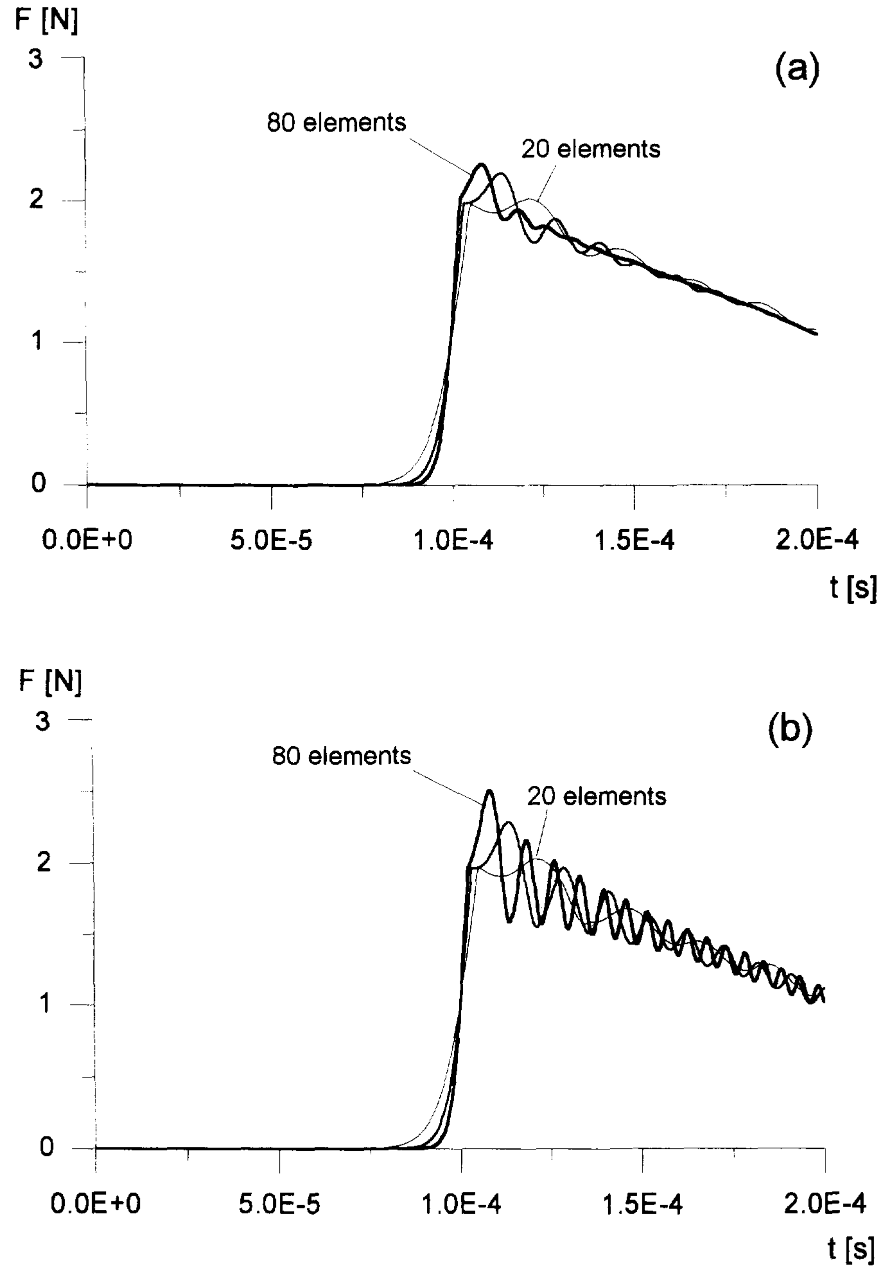


Fig. 4. Evolution of the built-in end reaction for the three meshes. Dynamic integration scheme: (a) $\beta = \gamma = 0.6$, (b) $\beta = \gamma = 0.5$.

$$\bar{\boldsymbol{\Phi}} \equiv \bar{\mathbf{N}}^{\mathrm{T}}\bar{\boldsymbol{\sigma}} - (\bar{\mathbf{h}}+\bar{\mathbf{c}})\bar{\boldsymbol{\lambda}} - \bar{\mathbf{k}} \leqslant \mathbf{0} \tag{55}$$

and compatibility (30) reads:

$$\bar{\mathbf{C}}\,\Delta\bar{\mathbf{u}} = \Delta\bar{\mathbf{e}} + \bar{\mathbf{N}}\,\Delta\bar{\boldsymbol{\lambda}} \tag{56}$$

Substituting (56) and (28) into the dynamic equilibrium equation (27), solving for $\bar{\mathbf{u}}$ the resulting equation and substituting into (55), one obtains the following formulation of the step problem

$$-\bar{\mathbf{\Phi}} \equiv \mathbf{A}\,\Delta\bar{\boldsymbol{\lambda}}+(\bar{\mathbf{k}}+(\bar{\mathbf{h}}+\bar{\mathbf{c}})\bar{\boldsymbol{\lambda}}_n-\bar{\mathbf{N}}^{\mathrm{T}}(\bar{\boldsymbol{\sigma}}_n+\Delta\bar{\boldsymbol{\sigma}}_E)) \geqslant \mathbf{0},\quad \Delta\bar{\boldsymbol{\lambda}} \geqslant \mathbf{0},\quad \bar{\mathbf{\Phi}}^{\mathrm{T}}\,\Delta\bar{\boldsymbol{\lambda}} = 0 \tag{57}$$

where the following quantities have been introduced:

$$\mathbf{Z} \equiv \bar{\mathbf{E}}\bar{\mathbf{C}}(\mathbf{K}_E+\bar{\mathbf{L}})^{-1}\bar{\mathbf{C}}^{\mathrm{T}}\bar{\mathbf{E}}-\bar{\mathbf{E}};\quad \mathbf{A} \equiv -\bar{\mathbf{N}}^{\mathrm{T}}\mathbf{Z}\bar{\mathbf{N}}+\bar{\mathbf{h}}+\bar{\mathbf{c}};\quad \Delta\bar{\boldsymbol{\sigma}}_E \equiv \bar{\mathbf{E}}\bar{\mathbf{C}}(\mathbf{K}_E+\bar{\mathbf{L}})^{-1}\,\Delta\bar{\mathbf{F}} \tag{58}$$

Problem (57) is a standard linear complementarity problem, with symmetric matrix of coefficients **A**. Therefore, one can conclude that the solution of the one-dimensional problem considered is unique in terms of $\Delta\bar{\boldsymbol{\lambda}}$ *if and only if* matrix **A** is positive definite. The necessary and sufficient uniqueness condition for $\Delta\bar{\boldsymbol{\lambda}}$ is a sufficient uniqueness condition for $\Delta\bar{\mathbf{p}}$, $\Delta\bar{\boldsymbol{\sigma}}$, $\Delta\bar{\mathbf{e}}$ and $\Delta\bar{\mathbf{u}}$.

This condition is in general difficult to apply since matrix **A** is a dense, non-diagonal matrix, with dimensions equal to the global number of nodes n_n. The computation of **A** implies the inversion of a matrix at the structural level. To obtain from the Necessary and Sufficient uniqueness Condition a critical time-step amplitude Δt_{NSC} (such that for $\Delta t < \Delta t_{NSC}$ uniqueness is guaranteed and for $\Delta t \geqslant \Delta t_{NSC}$ non uniqueness occurs), one has to explicitly compute all principal minors a_i and impose the condition $a_i > 0$ for $i = 1, \ldots n_n$.

This has been done for the bar considered, discretized by only two elements of length l^e. In this simple case all calculations can be carried out explicitly and the a_i can be computed as functions of Δt, of the softening coefficient h and of the gradient coefficient c.

Assuming in (58) $\bar{\mathbf{L}}$ as the algorithmic lumped mass matrix, the resulting critical value Δt_{NSC} of Δt is plotted in Fig. 5a as a function of c for varying h. In this figure all quantities are normalized as follows: $\Delta\tilde{t} = \Delta t/T_0$ ($T_0 = 2\pi/\omega_0 = (2\pi l/\sqrt{2-\sqrt{2}})\sqrt{\rho/E}$ being the first eigenperiod of the discretized structure), $\tilde{c} = c/l^{e2}E$, $\tilde{h} = h/E$. Notice that $\tilde{c} \geqslant 0$ and $\tilde{h} \in [-1, 0]$ for softening behaviour excluding snap-back at the material level. As expected, the critical time step $\Delta\tilde{t}_{NSC}$ decreases for increasing softening. The regularization effect of the gradient term appears: the time step ensuring uniqueness increases with c and this effect is amplified upon mesh refinement (decreasing element size l^e). For all values of $\tilde{h}$ there is a range of $\tilde{c}$ for which $\Delta t_{NSC} = 0$, i.e. for which non uniqueness of $\Delta\bar{\boldsymbol{\lambda}}$ exists for every time step. However, multiplicity of $\Delta\bar{\boldsymbol{\lambda}}$ does not necessarily imply multiplicity of solutions in terms of displacements and stresses. We will come back to this point later.

Figure 5b shows the same kind of results computed with the consistent mass matrix. The qualitative behaviour is exactly the same, but the critical values of Δt are smaller than the corresponding ones of Fig. 5a, computed with the lumped mass matrix; this difference tends to vanish for large $\tilde{c}$. Even if the example is very simple, this indicates the influence of the mass distribution on the uniqueness condition.

For the same problem, we have also computed the critical time step Δt_{SC} (eqn 50) resulting from the sufficient uniqueness condition established in Section 4.

This condition rests on the validity of inequality (44), proved under the hypothesis that the Gauss points used to model stresses and strains coincide with those used for numerical integration of all quantities. In this example only one Gauss point per element has been used for stress and strain interpolations. The one point, reduced integration of the softening matrix, defined in eqn (32c), gives a value $\bar{\mathbf{h}}^e_R$ which differs from the analytically integrated one $\bar{\mathbf{h}}^e$, defined in box 1, and used in the analyses. However, one can easily prove that:

$$\bar{\mathbf{h}}^e+\bar{\mathbf{c}}^e = \bar{\mathbf{h}}^e_R+\bar{\mathbf{c}}^e_R;\quad \text{where } \bar{\mathbf{h}}^e_R \equiv \frac{\Omega^e h}{4}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix};\quad \bar{\mathbf{c}}^e_R \equiv \left(\frac{c+hl^{e2}/12}{c}\right)\bar{\mathbf{c}}^e \tag{59}$$

where $\bar{\mathbf{c}}_R$ is positive semidefinite for:

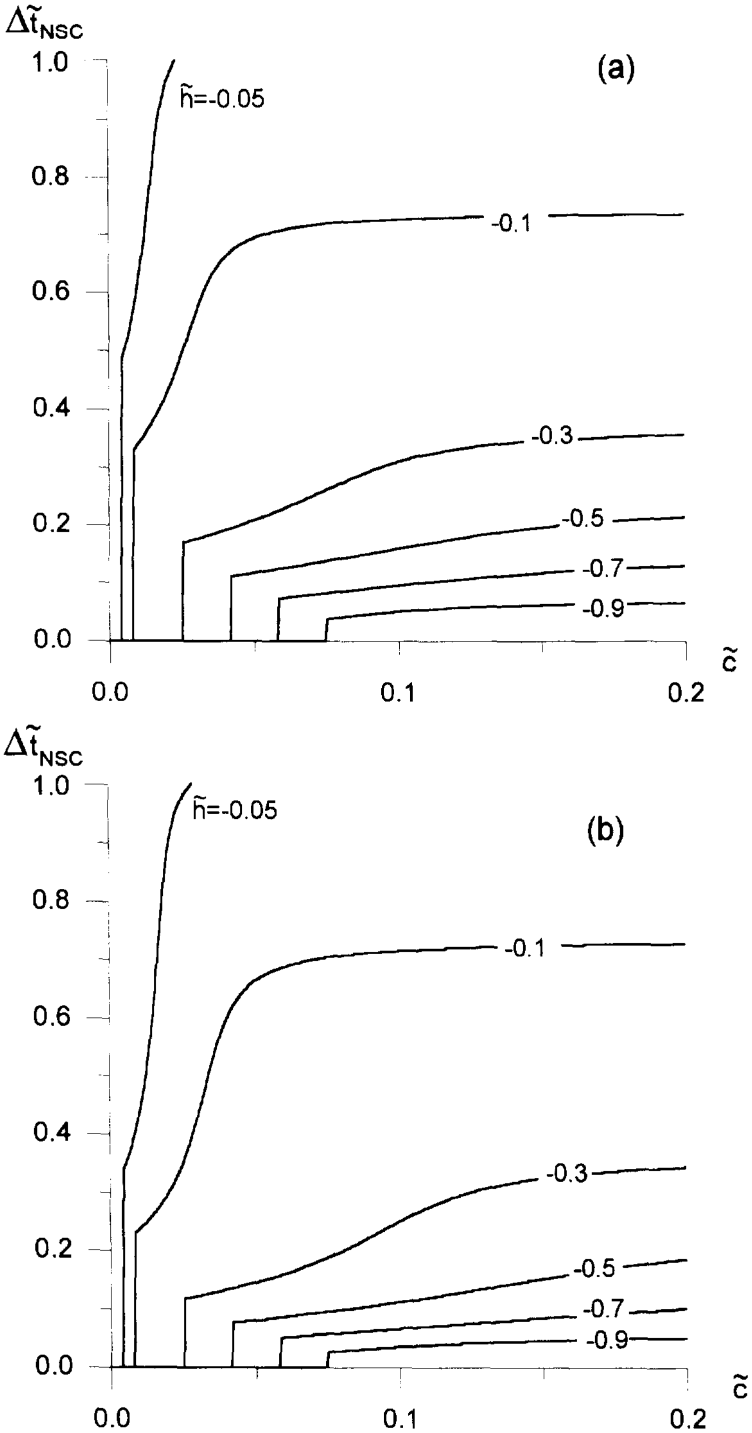


Fig. 5. Normalized critical time step amplitude of the necessary and sufficient uniqueness condition versus normalized diffusion parameter $\tilde{c}$ for varying normalized softening parameter $\tilde{h}$; (a) lumped mass matrix, (b) consistent mass matrix.

$$\tilde{c} \geqslant -\tilde{h}/12 \tag{60}$$

Hence, if $\tilde{c} \geqslant -\tilde{h}/12$, it is possible to replace in (43) matrix $\bar{\mathbf{h}}+\bar{\mathbf{c}}$ with matrix $\bar{\mathbf{h}}_R$ preserving the inequality and thus make use of inequality (44) in order to prove the sufficient uniqueness condition.

Proposition 3 has been applied with the above restriction (60). From the condition of positive semidefiniteness of $\zeta/(\zeta+1)\bar{\mathbf{E}}+\bar{\mathbf{h}}$ one obtains $\zeta/(\zeta+1) \geqslant -h/E = -\tilde{h}$, hence

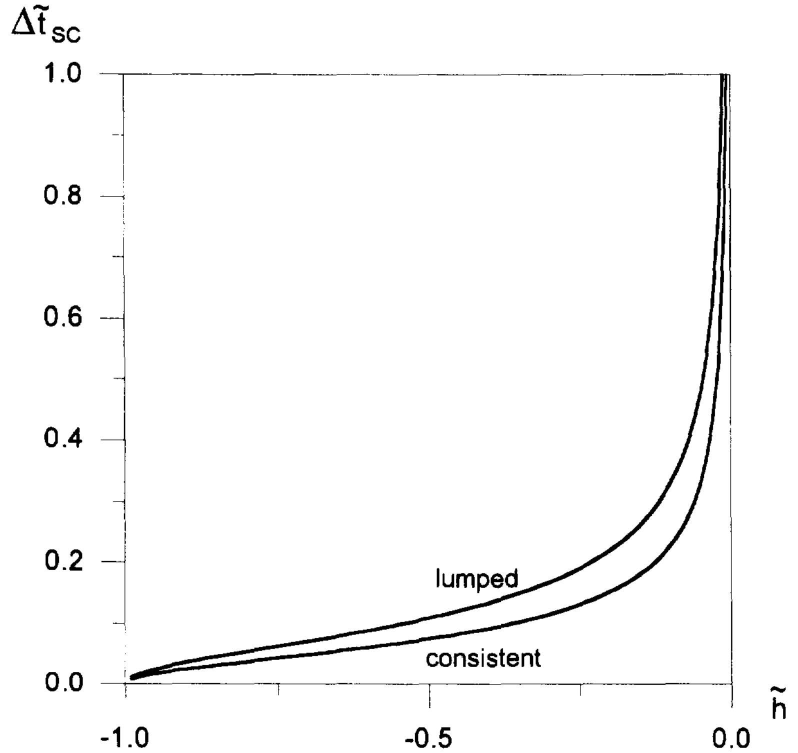


Fig. 6. Normalized critical time step amplitude of the sufficient uniqueness condition vs normalized softening parameter $\tilde{h}$.

$\zeta \geqslant (-\tilde{h})/(1+\tilde{h})$. By taking $\zeta = (-\tilde{h})/(1+\tilde{h})$, $\beta = \gamma = 0.6$, relation (50), which defines the critical time step, gives:

$$\Delta\tilde{t} = \frac{\Delta t}{T_0} < \frac{5}{6\pi}\frac{\omega_0}{\omega_M}\sqrt{\frac{1+\tilde{h}}{-\tilde{h}}} \equiv \Delta\tilde{t}_{SC} \tag{61}$$

For the two-elements discretization, if one considers the lumped mass matrix it results $\omega_0^2/\omega_M^2 = (2-\sqrt{2})/(2+\sqrt{2})$, while with the consistent mass matrix it results: $\omega_0^2/\omega_M^2 = (5/3-\sqrt{2})/(5/3+\sqrt{2})$. The critical time step $\Delta\tilde{t}_{SC}$, corresponding to the two different mass matrices, is plotted in Fig. 6 as a function of $\tilde{h}$. By comparing the critical time step of the sufficient condition (Fig. 6) with that of the necessary and sufficient conditions (Fig. 5), one can observe that, for a given $\tilde{h}$, $\Delta\tilde{t}_{SC}$ results to be coincident with $\Delta\tilde{t}_{NSC}$ corresponding to the break point of the curves of Fig. 5.

For $\Delta\tilde{t} < \Delta\tilde{t}_{SC}$ and $\tilde{c} \geqslant -\tilde{h}/12$, uniqueness in terms of $\Delta\bar{\mathbf{u}}$ and $\Delta\bar{\mathbf{p}}$ is guaranteed. To have uniqueness also of $\Delta\bar{\boldsymbol{\lambda}}$ one must have uniqueness of the linear complementarity problem giving $\Delta\bar{\boldsymbol{\lambda}}$ for assigned $\Delta\bar{\mathbf{u}}$ (i.e. of the linear complementarity problem of the corrector phase defined by eqns (39), not to be confused with the global linear complementarity problem (57)). Namely, for a given $\Delta\bar{\mathbf{u}}$ the solution is unique in $\Delta\bar{\boldsymbol{\lambda}}$ if and only if matrix $\mathbf{D}^e$ (see box 1) is positive definite. From this condition one obtains the following limitation on the amount of softening:

$$\tilde{c} > -\tilde{h}/12 \tag{62}$$

This means that if the gradient term c does not exist, or is too little, one has multiplicity of solutions in terms of $\Delta\bar{\boldsymbol{\lambda}}$ for any step amplitude.

This finding is in agreement with the global necessary and sufficient uniqueness condition. In fact the critical situation given by relation (62), i.e. $\tilde{c} = -\tilde{h}/12$, corresponds to the break points of the curves in Fig. 5a, b. For $\tilde{c} \leqslant -\tilde{h}/12$, i.e. on the left of the break

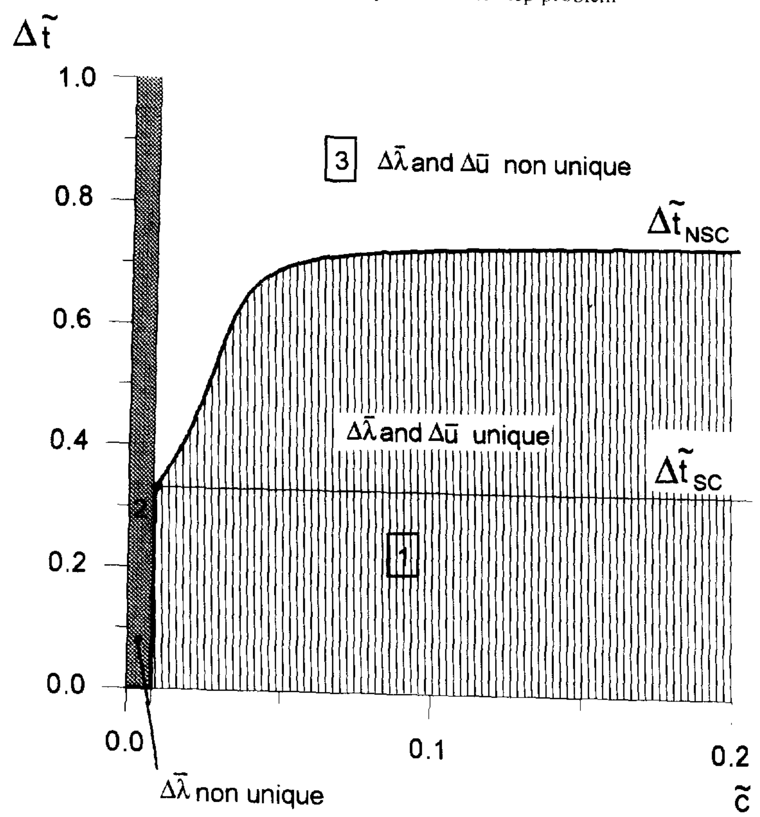


Fig. 7. Uniqueness and non uniqueness fields for $\tilde{h} = -0.1$ and lumped mass matrix.

point, uniqueness cannot be found for any value of $\Delta\tilde{t}$. This situation corresponds to the field 2 in Fig. 7, where the results concerning uniqueness have been summarized for $\tilde{h} = -0.1$. In the region 3 of Fig. 7 one has non uniqueness in $\Delta\bar{\lambda}$ and also in $\Delta\bar{\mathbf{u}}$. The nonuniqueness in $\Delta\bar{\lambda}$ follows from the fact that $\Delta\tilde{t}$ is greater than $\Delta\tilde{t}_{NSC}$. Moreover, inequality (62) being satisfied being in this field, multiplicity of $\Delta\bar{\lambda}$ is possible only if there is multiplicity in $\Delta\bar{\mathbf{u}}$. In region 1 of the same figure, uniqueness of $\Delta\bar{\lambda}$ and $\Delta\bar{\mathbf{u}}$ is guaranteed.

Remarks

1. For the problem considered in Section 5.1, one can easily compute a lower bound on the critical time step which ensures uniqueness replacing in (61) the highest frequency of the discretized system with the highest frequency of the single finite element. In this case, for one element, it results: $\omega_M = (2/l^e)\sqrt{E/\rho}$ and, with the data of Fig. 1, one obtains $\Delta t_{SC} = 2.5 \times 10^{-6}\, l^e$, l^e being the length of the finite element. For the finer mesh used in the analyses (80 elements), the critical time step is then $\Delta t_{SC} = 3.125 \times 10^{-6}$ s. The time step adopted in the analyses, $\Delta t = 5 \times 10^{-7}$ s, is therefore in the range where uniqueness of $\Delta\bar{\mathbf{u}}$ and $\Delta\bar{\mathbf{p}}$ is guaranteed.

2. From the data of Fig. 1, the conditions of applicability of proposition 3, eqn (60), and of uniqueness in $\Delta\bar{\lambda}$ of the corrector step eqn (62), give: $l^{e^2} < -12c/h = 300$ mm, which is fulfilled by all the meshes considered.

6. CONCLUSIONS

In the presence of softening materials the dynamic evolution problem can become ill-posed. As a result in step-by-step finite elements analyses some problems can arise, among them pathological mesh dependence and possible non uniqueness of the finite step solution.

This paper focused on the dynamic elastoplastic softening problem regularized by the introduction of the second order gradient of the equivalent plastic strains in the yield function. The Mises and Drucker-Prager models with linear isotropic softening have been considered.

The results of the present study can be summarized as follows.

(a) A variational formulation of the dynamic finite-step problem has been presented. The additional boundary conditions on plastic multipliers required by the considered gradient model are part of the solution of the optimization problem, therefore they do not intervene in the definition of the admissible set of plastic multipliers.

(b) A consistent finite element formulation in terms of generalized variables of the finite step problem has been obtained by discretizing all fields appearing in the functional of the variational property and by expressing its optimality conditions.

(c) Sufficient conditions for uniqueness of the solution of the dynamic finite-step problem have been established. In particular a bound on the time-step amplitude ensuring uniqueness has been derived. This bound depends on the amount of softening and on inertia and stiffness properties through the maximum eigenfrequency.

(d) For a one-dimensional problem a necessary and sufficient condition for the uniqueness of the dynamic finite-step problem has also been presented. This condition highlights the regularization effect of the gradient term. Namely, the admissible time step amplitude ensuring uniqueness increases as the coefficient of the gradient term increases.

The proposed formulation has been tested with reference to the same problem discussed in Sluys *et al.* (1993). The results obtained are almost mesh independent and the material internal length introduced by the gradient approach governs the amplitude of the localized zone regardless the space discretization. The generalized variable approach seems therefore well suited for gradient-dependent plasticity in the dynamic range.

The sufficient uniqueness condition has also been applied to the same problem and a significant bound on the time step amplitude to be adopted in the analyses has been computed.

Application of two dimensional problems of generalized variables finite elements with nonlocal (gradient-dependent) material models can be found in Comi and Perego (1996) for static and in Comi and Corigliano (1995) for dynamics.

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APPENDIX A—PROOF OF INEQUALITY (44)

The gradient of the generalized yield function $\bar{\boldsymbol{\varphi}}$ with respect to the generalized stress vector can be expressed in the following way:

$$\frac{\partial \bar{\boldsymbol{\varphi}}^{\mathrm{T}}}{\partial \bar{\boldsymbol{\sigma}}} = \int_{\Omega} \frac{\partial}{\partial \bar{\boldsymbol{\sigma}}}(\varphi(\boldsymbol{\sigma})\mathbf{N}_{\lambda})\,\mathrm{d}\Omega = \int_{\Omega} \mathbf{N}_{\sigma}^{\mathrm{T}} \frac{\partial \varphi}{\partial \boldsymbol{\sigma}} \mathbf{N}_{\lambda}\,\mathrm{d}\Omega = \sum_{g=1}^{n_{Gp}} W^{g} \mathbf{N}_{\sigma}^{\mathrm{T}}(\mathbf{x}^{g}) \frac{\partial \varphi}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}(\mathbf{x}^{g}))\mathbf{N}_{\lambda}(\mathbf{x}^{g}) \tag{A.1}$$

In the above inequalities use has been made of relation (32b) defining $\bar{\boldsymbol{\varphi}}$, of the interpolation for stresses (22c) and of the Gauss integration (34). With $\mathbf{N}_{\sigma}$ defined by (36) and (35), where the set of *nGp* Gauss points is the same as that used in (A.1), we find:

$$\frac{\partial \bar{\boldsymbol{\varphi}}^{\mathrm{T}}}{\partial \bar{\boldsymbol{\sigma}}} = \begin{bmatrix} \partial_{\sigma}\varphi(\boldsymbol{\sigma}(\mathbf{x}^{1}))\mathbf{N}_{\lambda}(\mathbf{x}^{1}) \\ \cdots \\ \partial_{\sigma}\varphi(\boldsymbol{\sigma}(\mathbf{x}^{g}))\mathbf{N}_{\lambda}(\mathbf{x}^{g}) \\ \cdots \\ \partial_{\sigma}\varphi(\boldsymbol{\sigma}(\mathbf{x}^{n_{Gp}}))\mathbf{N}_{\lambda}(\mathbf{x}^{n_{Gp}}) \end{bmatrix} \tag{A.2}$$

Recall the definitions (33c) of $\Delta\bar{\mathbf{p}}$ and (37) of matrix $\hat{\mathbf{h}}$ and make use of (A.2), to obtain:

$$\Delta\bar{\mathbf{p}}_1^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\mathbf{p}}_2 = \Delta\bar{\boldsymbol{\lambda}}_1^{\mathrm{T}}\left[\sum_{g=1}^{n_{G}} \Theta h(\mathbf{x}^g)\,W^g\mathbf{N}_\lambda^{\mathrm{T}}(\mathbf{x}^g)\frac{\partial\varphi}{\partial\boldsymbol{\sigma}^{\mathrm{T}}}(\boldsymbol{\sigma}_1(\mathbf{x}^g))\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_2(\mathbf{x}^g))\mathbf{N}_\lambda(\mathbf{x}^g)\right]\Delta\bar{\boldsymbol{\lambda}}_2 \tag{A.3}$$

where 1 and 2 denote two different solutions.

It can now be observed that for the Mises effective stress $\varphi(\sigma) = \sqrt{3J_2}$, or the Drucker-Prager effective stress $\varphi(\sigma) = \sqrt{3J_2} + \alpha I_1/3$, the following relations hold:

$$\frac{\partial\varphi}{\partial\boldsymbol{\sigma}^{\mathrm{T}}}(\boldsymbol{\sigma}_1(\mathbf{x}))\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_2(\mathbf{x})) \leqslant \frac{1}{\Theta} = \frac{\partial\varphi}{\partial\boldsymbol{\sigma}^{\mathrm{T}}}(\boldsymbol{\sigma}_1(\mathbf{x}))\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_1(\mathbf{x})) = \frac{\partial\varphi}{\partial\boldsymbol{\sigma}^{\mathrm{T}}}(\boldsymbol{\sigma}_2(\mathbf{x}))\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_2(\mathbf{x})) \tag{A.4}$$

Relations (A.4) express the fact that the norm of the vector normal to the yield surface is constant for any σ and that the scalar product of the vectors normal to the yield surface at $\sigma = \sigma_1$ and $\sigma = \sigma_2$ cannot exceed the product of their norms.

Equation (A.3) written with $1 = 2 = {}'$, $1 = 2 = {}''$ and $1 = {}'$, $2 = {}''$ gives rise to the relations:

$$\Delta\bar{\mathbf{p}}'^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\mathbf{p}}' = \Delta\bar{\boldsymbol{\lambda}}'^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\boldsymbol{\lambda}}'; \quad \Delta\bar{\mathbf{p}}''^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\mathbf{p}}'' = \Delta\bar{\boldsymbol{\lambda}}''^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\boldsymbol{\lambda}}''; \tag{A.5a, b}$$

$$\Delta\bar{\mathbf{p}}'^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\mathbf{p}}'' \geqslant \Delta\bar{\boldsymbol{\lambda}}'^{\mathrm{T}}\bar{\mathbf{h}}\,\Delta\bar{\boldsymbol{\lambda}}'' \tag{A.5c}$$

where the definition (32c) of $\bar{\mathbf{h}}$, with $h < 0$, and the Gaussian integration rule (34) have been used. Combining relations (A.5) one obtains:

$$(\Delta\bar{\mathbf{p}}' - \Delta\bar{\mathbf{p}}'')^{\mathrm{T}}\bar{\mathbf{h}}(\Delta\bar{\mathbf{p}}' - \Delta\bar{\mathbf{p}}'') \leqslant (\Delta\bar{\boldsymbol{\lambda}}' - \Delta\bar{\boldsymbol{\lambda}}'')^{\mathrm{T}}\bar{\mathbf{h}}(\Delta\bar{\boldsymbol{\lambda}}' - \Delta\bar{\boldsymbol{\lambda}}'') \tag{A.6}$$

which proves inequality (44).